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PROJECT REPORT

Single Photon Sources and Beam Splitters

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Project Statement :

1. To implement a model for a single photon source using random number generators that emits photons 1] uniformly 2] exponentially 3] logarithmically distributed in time.
2. Implementing a model for a 50-50 beam splitters using random number generators and using it to validate the single photon source by plotting the second order correlation $g_2(\tau)$ between the outputs.
3. Deriving an expression for the beam splitter operator using the states of the quantum harmonic oscillator as eigenbasis and using it to analyse the outputs for 1] two single photon source input 2] one n-photon input 3] coherent state input.

Single Photon Source

A single photon source is a light source that emits light in the form of discrete photons. The interval between the emission of two photons can vary with time. The emission distribution depends on how the probability of a photon being emitted varies with time. We consider the following distributions :-

1) Uniform : 50% probability of emitting a photon at any instant of time.
2) Exponential : Probability of emitting a photon varies with time as $e^{-t/\tau}$.
3) Logarithmic : Probability of emitting a photon varies with time as $1-\log(1+t/\tau)$

The source is implemented by digitizing the output of a random number generator - a number between 0 and 1 is chosen randomly from a distribution and then digitized as 1 if it is greater than 0.5 ~~or~~ as 0 else.

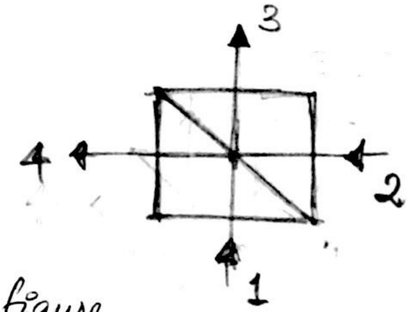
1] To get the uniform distribution, we choose the number out of a uniform distribution. The probability of getting a photon i.e a 1 at any instant then equals the probability of getting a number greater than 0.5 which is equal to 50 %.

The generated array thus represents a set of measurements taken at discrete time instants i over a period of time N, where N is the length of the array.

For the exponential distribution, note that the probability of getting a photon (1) at an instant i is $e^{-i/N}$. So to get the distribution at each instant i we toss a biased coin whose probability of giving heads i.e 1 is $e^{-i/N}$. If required, we can create the un-digitized distribution by choosing a number randomly from 0.5 to 1 if the toss gave a 1, or from 0 to 0.5 if it gave a 0. Digitizing the distribution will not affect the final array output

[Please refer to Figures 1, 2 and 3 for the generated, digitized distributions and Snippets 1, 2 and 3 for the code that generates them.]

Beam Splitter.



A beam splitter is an optical instrument that is capable of splitting an incoming light beam into two outputs. It is also capable of splitting two input beams simultaneously. In the adjacent figure, 1 and 2 represent the input beams while 3 and 4 represent the outputs. We assume that the beam splitter is lossless, i.e. it conserves energy. If $\vec{E}_1$ and $\vec{E}_2$ represent input electric fields while $\vec{E}_3$ and $\vec{E}_4$ represent output electric fields, we can write

$$\vec{E}_3 = \underbrace{T_{13}\vec{E}_1}_{\text{(transmitted)}} + \underbrace{R_{23}\vec{E}_2}_{\text{(reflected)}} \quad \text{and} \quad \vec{E}_4 = \underbrace{T_{24}\vec{E}_2}_{\text{(transmitted)}} + \underbrace{R_{14}\vec{E}_1}_{\text{(reflected)}}$$

equivalently $\begin{bmatrix} \vec{E}_3 \\ \vec{E}_4 \end{bmatrix} = \begin{bmatrix} T_{13} & R_{23} \\ R_{14} & T_{24} \end{bmatrix} \begin{bmatrix} \vec{E}_1 \\ \vec{E}_2 \end{bmatrix}$. For conservation of energy,

$$|\vec{E}_3|^2 + |\vec{E}_4|^2 = |\vec{E}_1|^2 + |\vec{E}_2|^2$$

Now let $T_{13} = t_{13}e^{i\phi_{13}}$, $T_{24} = t_{24}e^{i\phi_{24}}$, $R_{14} = r_{14}e^{i\phi_{14}}$, and $R_{23} = r_{23}e^{i\phi_{23}}$.

we get from energy conservation:

$$\left(|T_{13}|^2 + |R_{14}|^2\right)|\vec{E}_1|^2 + \left(|T_{24}|^2 + |R_{14}|^2\right)|\vec{E}_2|^2 + \left(R_{14}T_{24}^* + T_{13}R_{23}^*\right)E_2^*E_1 + \left(R_{14}^*T_{24} + T_{13}^*R_{23}\right)E_1E_2^* = |\vec{E}_1|^2 + |\vec{E}_2|^2$$

a simple solution to the above equation is

$$|T_{13}|^2 + |R_{14}|^2 = 1, \quad |T_{24}|^2 + |R_{14}|^2 = 1, \quad R_{14}T_{24}^* + T_{13}R_{23}^* = 0$$

substituting the polar forms, we get

$t_{13} r_{23} e^{j(\phi_{23}-\phi_{13})} + t_{24} r_{14} e^{j(\phi_{24}-\phi_{14})} = 0 \Rightarrow \frac{t_{13} r_{23}}{t_{24} r_{14}} = -e^{j(\phi_{24}-\phi_{23}+\phi_{13}-\phi_{14})}$

Note that $t_{13}, r_{23}, t_{24}, r_{14} \in R^+$, so we must have $\phi_{24}-\phi_{23}+\phi_{13}-\phi_{14} = \pi(2n+1)$ as a consequence of $t_{13}^2 + r_{14}^2 = 1$ and $t_{24}^2 + r_{23}^2 = 1$. For simplicity we $n \in Z$ can assume $\phi_{24} = \phi_{13} = 0$ and $\phi_{23} = \phi_{14} = \pi/2$. This also gives $t_{13} r_{23} = t_{24} r_{14}$ which implies $t_{13} = t_{24} = t$ and $r_{23} = r_{14} = r$. So our matrix equation is finally

$$\begin{bmatrix} \vec{E}_3 \\ \vec{E}_4 \end{bmatrix} = \begin{bmatrix} t & ir \\ ir & t \end{bmatrix} \begin{bmatrix} \vec{E}_1 \\ \vec{E}_2 \end{bmatrix} \quad \text{where } t^2 + r^2 = 1.$$

We can thus define
t = transmission coefficient
r = reflection coefficient.

In order to verify the single photon source, we can construct a beam splitter by using a random number generator to decide which output each photon exits from. If the reflectance (r^2) of the beam splitter is R, then each photon has a probability R of being reflected by the beam splitter. So we use the random number generator to uniformly pick a number between 0 and 1. If it is less than R, we reflect the photon, otherwise it is transmitted. For a 50:50 beam splitter, $R = 0.5$.

[Please refer to Figure 4 for sample output generated using a uniform distribution. Snippets 4 and 5 contain code used to generate and plot the output]

Correlation

In order to quantifiably measure the similiarity between the two outputs of the beam splitter as a function of the displacement in time relative to one another, we calculate the second-order correlation between the two outputs. It is defined as follows

$$g_2^{(2)}(\tau) = \frac{\langle I_3(t)\, I_4(t+\tau) \rangle}{\langle I_3(t) \rangle \langle I_4(t+\tau) \rangle}$$

where I_3 and I_4 are the intensities observed at output terminals 3 and 4 respectively.

As intensity of an EM wave is directly proportional to the electric field squared, we can write

$$g_2^2(\tau) = \frac{\langle E_3^2(t)\, E_4^2(t+\tau) \rangle}{\langle E_3^2(t) \rangle \langle E_4^2(t+\tau) \rangle}$$

In terms of the annihilation and creation operators of the quantum oscillator, we have the electric field operator

$\hat{E} = E_0 (a e^{i\phi} + a^+ e^{-i\phi})$ where $\phi = \vec{k} \cdot \vec{r} - \omega t$ is the phase.

Substituting into the previous expression and keeping only the terms with non-zero expectation values, we get

$$g_2^2(\tau) = \frac{\langle a_3^+(t)\, a_4^+(t+\tau)\, a_4(t+\tau)\, a_3(t) \rangle}{\langle a_3^+(t) \rangle \langle a_4^+(t+\tau) \rangle}$$

which further simplifies to

$$g_2^2(\tau) = \frac{\langle N_3(t)\, N_4(t+\tau) \rangle}{\langle N_3(t) \rangle \langle N_4(t+\tau) \rangle}.$$ where $N = a^+ a$ is the number operator.

The implemented model outputs the number of photons detected at each output - so we can directly use above expression to calculate correlation. Since our implementation is in the discrete time domain, we equivalently have

$$g_2^2[\tau] = \frac{T \sum_{t=1}^{T} N_3[t]\, N_4[(t+\tau) \bmod T]}{\left(\sum_{t=1}^{T} N_3[t] \right) \left(\sum_{t=1}^{T} N_4[t] \right)}$$

(the modulus is to prevent array overflow)

T is the length of the array, the duration for which measurements were made.

For a single-photon source, we must have that

$g_2^2(0) = 0$ as the photon cannot be at both outputs at the same time

$\Rightarrow N_3[t]\, N_4[t] = 0 \;\; \forall\, t \Rightarrow g_2^2(0) = 0.$

[Please refer to Figure 5 for sample correlation calculated for outputs generated using a uniform distribution input. Snippet 6 contains the code used to calculate the correlation]

Derivation of the Beam Splitter Operator.

Consider m photons entering from input 1 and n photons simultaneously entering the beam splitter from terminal 2. We represent this state as $|m\,n\rangle$. If the total number of photons entering is fixed at N, we have $|00\rangle, |01\rangle \ldots\ |N0\rangle = \frac{N(N-1)}{2}$ possible input states

We represent these states as an $N^2 \times N^2$ matrix. If the beam splitter operator is represented by B, then elements of the beam splitter are given by $\langle pq|B|mn\rangle$.

Firstly we derive a relation between the annihilation operators a_1, a_2 of the input fields and that of the output fields (a_3, a_4)

The electric field operator $\hat{E}$ is given as $\hat{E} = (ae^{i\phi} + a^{+}e^{i\phi})E_0$.

Considering the complex analogue, $\hat{\mathcal{E}}_s = \mathcal{E}_0 a e^{i\phi}$. As shown earlier,

$$\begin{bmatrix} \vec{E}_3 \\ \vec{E}_4 \end{bmatrix} = \begin{bmatrix} t & ir \\ ir & t \end{bmatrix} \begin{bmatrix} \vec{E}_2 \\ \vec{E}_1 \end{bmatrix}$$

In terms of the complex operator $\hat{\mathcal{E}}_s$,

$$\begin{bmatrix} \hat{\mathcal{E}}_{s_3}\psi_3 \\ \hat{\mathcal{E}}_{s_4}\psi_4 \end{bmatrix} = \begin{bmatrix} t & ir \\ ir & t \end{bmatrix} \begin{bmatrix} \hat{\mathcal{E}}_{s_1}\psi_1 \\ \hat{\mathcal{E}}_{s_2}\psi_2 \end{bmatrix}$$

which gives the following relations

$$a_3 e^{i\phi_3}\psi_3 = t a_1 e^{i\phi_1}\psi_1 + i r a_2 e^{i\phi_2}\psi_2$$
$$a_4 e^{i\phi_4}\psi_4 = t a_2 e^{i\phi_2}\psi_2 + i r a_1 e^{i\phi_1}\psi_1$$

These equations are valid for all arbitrary phases. so we can write

$$a_3\psi_3 = t a_1\psi_1 + i r a_2\psi_2$$
$$a_4\psi_4 = t a_2\psi_2 + i r a_1\psi_1$$

As per our state notation, $|\psi_{in}\rangle = |\psi_1\psi_2\rangle$ and $|\psi_{out}\rangle = |\psi_3\psi_4\rangle$

The operator a_1 has no effect on ψ_2 and a_2 has no effect on ψ_1. So

$$a_3\psi_3 = (t a_1 + i r a_2)|\psi_{in}\rangle \quad -①$$
$$a_4\psi_4 = (t a_2 + i r a_1)|\psi_{in}\rangle \quad -②$$

From $t①-ir②$ and $t②-ir①$ we get

$$(t a_3 - i r a_4)|\psi_{out}\rangle = a_1|\psi_{in}\rangle$$
$$(t a_4 - i r a_3)|\psi_{out}\rangle = a_2|\psi_{in}\rangle$$

Taking adjoint on both sides,

$$(t a_3^{\dagger} + i r a_4^{\dagger})|\psi_{out}\rangle = a_1^{\dagger}|\psi_{in}\rangle \text{ and } (t a_4^{\dagger} + i r a_3^{\dagger})|\psi_{out}\rangle = a_2^{\dagger}|\psi_{in}\rangle \quad -\mathrm{I}$$

Since B is an operator, we have $|\psi_{out}\rangle = B|\psi_{in}\rangle$. Let $|\psi_{in}\rangle = |mn\rangle$

We can write $|\psi_{in}\rangle = |mn\rangle = \dfrac{(a_1^\dagger)^m (a_2^\dagger)^n}{\sqrt{m!\,n!}}|00\rangle$ using $a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$ repeatedly

Using the transformation relations I derived previously,

$$|\psi_{out}\rangle = \frac{(ta_3^\dagger + ira_4^\dagger)^m (ta_4^\dagger + ira_3^\dagger)^n}{\sqrt{m!\,n!}}|00\rangle = B|mn\rangle$$

(Since $|\psi_{in}\rangle = |00\rangle$ must give $|\psi_{out}\rangle = |00\rangle$, we substitute it into I fuse)

Hence the element $\langle pq|B|mn\rangle$ of the operator is given by

$$\langle pq|\frac{(ta_3^\dagger + ira_4^\dagger)^m (ta_4^\dagger + ira_3^\dagger)^n}{\sqrt{m!\,n!}}|00\rangle$$

Writing the binomial expansion of the operator above expression becomes

$$\langle pq|B|mn\rangle = \frac{1}{\sqrt{m!\,n!}}\langle pq|\sum_{k=0}^{m}\sum_{l=0}^{n} {}^mC_k\, {}^nC_l\, t^{n+k-l}(ir)^{m+l-k}(a_3^\dagger)^{k+l}(a_4^\dagger)^{m+n-(k+l)}|00\rangle$$

$$= \frac{1}{\sqrt{m!\,n!}}\langle pq|\sum_{k=0}^{m}\sum_{l=0}^{n} {}^mC_k\, {}^nC_l\, t^{n+k-l}(ir)^{m+l-k}|k+l, m+n-(k+l)\rangle\sqrt{(k+l)!}\sqrt{(m+n-(k+l))!}$$

So $$\langle pq|B|mn\rangle = \sum_{k=0}^{m}\sum_{l=0}^{n} {}^mC_k\, {}^nC_l\, t^{n+k-l}(ir)^{m+l-k}\langle pq|k+l, m+n-k-l\rangle\sqrt{\frac{(k+l)!\,(m+n-k-l)!}{m!\,n!}}$$

which implies $\langle pq|B|mn\rangle \neq 0$ if $p = k+l$ and $q = m+n-(k+l) = m+n-p$

$\therefore$ For $\langle pq|B|mn\rangle$ to be non-zero we require $p+q = m+n$, which is the same saying that the number of total photons must remain conserved.

Then for $p+q = m+n$,

$$\langle pq|B|mn\rangle = \sum_{\substack{k+l=p\\0\le k\le m\\0\le l\le n}} {}^mC_k\, {}^nC_l\, t^{n+k-l}(ir)^{m+l-k}\sqrt{\frac{p!\,q!}{m!\,n!}} = S_p^{mn}\sqrt{\frac{p!\,q!}{m!\,n!}}\,\delta_{p+q,m+n}$$

where $$S_p^{mn} = \sum_{\substack{k+l=p\\0\le k\le m\\0\le l\le n}} {}^mC_k\, {}^nC_l\, t^{n+k-l}(ir)^{m+l-k} = \sum_{\substack{k+l=p\\0\le k\le m\\0\le l\le n}} \left({}^mC_k\, t^k (ir)^{m-k}\right)\left({}^nC_l\, t^{n-l}(ir)^l\right)$$

Note that S_p^{mn} equals the coefficient of x^p in the expansion of $(tx+ir)^m(t+irx)^n$

So if we let $F(x) = (t+irx)^n(tx+ir)^m$, $S_p^{mn} = \dfrac{F^{(p)}(0)}{p!}$

where $F^{(n)}$ denotes nth derivative of F.

So the Beam Splitter operator is specified by the elements

$$B_{iN+j,\,pN+q} = \langle ij|B|pq\rangle = \sqrt{\frac{i!\,j!}{p!\,q!}}\,\delta_{i+j,\,p+q}\,\frac{F^{(i)}(0)}{i!}$$

where $B_{N^2 \times N^2}$ is the dimension of the truncated operator matrix.

[Please refer to Figures 6 and 7 for 4x4 and 9x9 versions of the beam splitter operator matrix. Snippet 7 contains the code used to generate the matrix.]

We then used the operator to find the output for the following input cases:

1. Single photon source input to one terminal.

The results agreed with the earlier outputs generated by randomly splitting the input [Refer Snippet 4]. $g_2^2(0)$ is observed to be 0.

[Please refer to Figure 8 for sample output generated using a uniform distribution input. Figure 9 shows the histograms. Figure 10 shows the correlation plot.]

[Please refer to Snippet 8 for the code that generates inputs for each case that is specified. Snippet 9 contains code that calculates the eigenfunction decomposition of the generated input. Snippet 10 contains code that converts the output wavefunction to an actual measurement i.e the output]

2. Two single photons as inputs at separate terminals

The Hong-Ou-Mandel effect is observed in the output. Both the photons exit from the same output terminal. $g_2^2(0)$ is observed to be 0.

[Please refer to Figure 11 for sample output generated with uniform input distributions. Figure 12 shows the histograms and Figure 13 shows the correlation plot]

Since $g_2^2(0) = 0$, we can conclusively say the photons never leave from different output terminals.

3. n-photon state input at one terminal

We observe that the input photons leave from either of the output terminals. The output state distribution is Poisson-like. $g_2^2(0) \neq 0$ as photons can simultaneously be observed at both output terminals in this case.

[Please refer Figure 14 for sample output generated using a uniform input distribution with $n=5$. Figure 15 shows the histograms while Figure 16 shows the correlation plot]

4. Coherent state input

The coherent state is represented as

$$\psi_N(x,t) = \sum_{n=0}^{\infty} C_{N,n} \exp\left\{-i\left(n+\tfrac{1}{2}\right)\omega t\right\} u_n(x) \quad , \quad |C_{N,n}|^2 = \frac{\exp(-N) \cdot N^n}{n!}$$

[Please refer to Snippet 11 for code that generates coherent state input]

The coherent state is an eigenfunction of the annihilation operator. We observe that the output states are also coherent states but with shifted means. The correlation is uniform throughout

[Please refer to Figure 17 for sample output generated using $N=4$ with $u_7(x)$ as the max. allowed eigenstate. Figure 18 shows the histograms and Figure 19 shows the correlation plot]